Calculer les limites suivantes d'intégrales :

a) $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$.

On sait que $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, d'où : $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{a \to +\infty} \left(\int_1^a \frac{1}{x^2}\,dx\right) = \lim_{a \to +\infty} \left[-\frac{1}{x}\right]_1^a = \lim_{a \to +\infty} \left(-\frac{1}{a} + 1\right) = +1$.

b) $\displaystyle\int_e^{+\infty} \frac{1}{x(\ln x)^2}\,dx$.

On sait que $(\ln x)' = \frac{1}{x}$ et $\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$, d'où : $\displaystyle\int_e^{+\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{a \to +\infty} \left(\int_e^a \frac{1}{x(\ln x)^2}\,dx\right) = \lim_{a \to +\infty} \left(\int_e^a \frac{\frac{1}{x}}{(\ln x)^2}\,dx\right)$,

$$\int_e^{+\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{a \to +\infty} \left[-\frac{1}{\ln x}\right]_e^a = \lim_{a \to +\infty} \left(-\frac{1}{\ln a} + \frac{1}{\ln e}\right) = \lim_{a \to +\infty} \left(1 - \frac{1}{\ln a}\right) = +1 .$$

c) $\displaystyle\int_1^{+\infty} \frac{1}{(1 + e^{-x})(1 - e^{-x})}\,dx$.

$$\frac{1}{(1 + e^{-x})(1 - e^{-x})} = \frac{1}{1 - e^{-2x}} = \frac{e^{2x}}{e^{2x}(1 - e^{-2x})} = \frac{e^{2x}}{e^{2x} - 1} .$$

On sait $(e^u)' = u'.e^u$, donc $(e^{2x})' = 2.e^{2x}$. Par ailleurs $(\ln u)' = \frac{u'}{u}$.

$$\int_1^{+\infty} \frac{1}{(1 + e^{-x})(1 - e^{-x})}\,dx = \lim_{a \to +\infty} \int_1^a \frac{e^{2x}}{e^{2x} - 1}\,dx = \lim_{a \to +\infty} \frac{1}{2}\int_1^a \frac{2e^{2x}}{e^{2x} - 1}\,dx = \frac{1}{2} \lim_{a \to +\infty} [\ln (e^{2x} - 1)]_1^a ,$$

$$\int_1^{+\infty} \frac{1}{(1 + e^{-x})(1 - e^{-x})}\,dx = \frac{1}{2}\left(\lim_{a \to +\infty} \ln (e^{2a} - 1)\right) - \frac{1}{2}\ln (e^2 - 1) = +\infty .$$

d) $\displaystyle\int_0^{+\infty} \frac{1}{(1 + e^{x})(1 + e^{-x})}\,dx$.

$$\frac{1}{(1 + e^{x})(1 + e^{-x})} = \frac{e^x}{(1 + e^x)(e^x + 1)} = \frac{e^x}{(e^x + 1)^2} .$$

On sait $(e^x)' = e^x$ et $\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$.

$$\int_0^{+\infty} \frac{1}{(1 + e^{x})(1 + e^{-x})}\,dx = \lim_{a \to +\infty} \int_0^a \frac{e^x}{(e^x + 1)^2}\,dx = \lim_{a \to +\infty} \left[-\frac{1}{e^x + 1}\right]_0^a = \lim_{a \to +\infty} \left(-\frac{1}{e^a + 1}\right) + \frac{1}{2} = +\frac{1}{2} .$$